Handout 7

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Notes:

- Parameters:
  - Non-logical parameters:
    - Relations
    - Terms (functions, constants, variables)
  - Logical parameters:
    - Connectives
    - Quantifiers
- Models:
  - (D, I)
  - $D =$ Domain:
    - Set of objects we're reasoning about
    - Example:
      - $\{alabama, texas, atlantis \ldots\}$
  - $I =$ Interpretation function:
    - Sends each non-logical parameter to an object of the appropriate sort, using D
      - $I(a) = a^I = alabama$
      - $I(B) = B^I = \{< california, nevada >,$
        $< atlantis, the\ state\ next\ to\ atlantis >, \ldots\}$
- Languages:
  - Formal language
    - = Set of well-formed formulas
      - E.g. $\forall x(R(x) \to P(x))$
  - Meta-language
    - = Language for talking about the formal language or logical system
      - E.g. $(D, I), I(B) \ldots$
- Variable assignments:
  - $g: Var \to D$
  - g_a^x = then function *g* sending *x* to *a*:
    - If $y \neq x$, then $g_a^x(y) = g(y)$
    - If $y = x$, then $g_a^x(y) = a$
- Interpretation of a term:
  - It's formal meaning
  - Interpretation of term t in model *M* under assignment g:
    - If t is a constant *c*, then $[[c]]_g^M = c^I$
    - If t is a variable *x*, then $[[x]]_g^M = g(g)$
    - If t is a complex term from applying a function, then $[[f(t_1, \ldots, t_n)]]_g^M =$ $f^I([[t_1]]_g^M, \ldots, [[t_n]]_g^M)$
- Truth under assignment *g:*
  - $M \vDash t_1 = t_2[g]$ iff $[[t_1]]_g^M = [[t_2]]_g^M$

- $M \vDash P(t_1, \dots, t_n)$ iff $P([[t_1]]_g^M, \dots, [[t_n]]_g^M)$
  - ...
  - $M \vDash \forall x\varphi$ iff for every $d \in D$: $M \vDash \varphi[g_d^x]$
  - $M \vDash \exists x\varphi$ iff for some $d \in D$: $M \vDash \varphi[g_d^x]$
- Free variables:
  - Variables not bound by a quantifier (at least once)
    - E.g. x and y in $(P(x) \wedge \forall x(R(x, y))$

- Validity:
  - φ is valid if $M \vDash \varphi[g]$ for all models M and all assignment functions g

Practice exercises:

- **Exercise 6.3.3:** 1 and 8-10
  - Here's a model over $D = \{a, b, c, d\}$:
    - $c^{\mathcal{I}} = c$ and $t^{\mathcal{I}} = a$.
    - $P^{\mathcal{I}} = \{a, b, c\}$
    - $f^{\mathcal{I}}(a) = b$, $f^{\mathcal{I}}(b) = b$, $f^{\mathcal{I}}(c) = a$, $f^{\mathcal{I}}(d) = a$
    - $Q^{\mathcal{I}} = \{\langle a, a, b\rangle, \langle b, a, d\rangle, \langle d, d, d\rangle, \langle c, a, d\rangle, \langle a, d, d\rangle\}$
  - Do the following hold?
    - 1. $\forall x(P(x) \vee P(f(x)))$
      8. $\exists x f(f(f(x))) = x$
      9. $\exists x \neg Q(x, x, x) \rightarrow f(x) = t$
      10. $\exists x(Q(x, x, x) \wedge Q(c, t, x))$
    -
- **Exercise 6.3.4:** 1 and 8-10
  - Here's another model over $D = \{a, b, c, d\}$:
    - $c^{\mathcal{I}'} = b$ and $t^{\mathcal{I}'} = d$.
    - $P^{\mathcal{I}'} = \{a, c, d\}$
    - $f^{\mathcal{I}'}(a) = f^{\mathcal{I}'}(b) = f^{\mathcal{I}'}(c) = d$, $f^{\mathcal{I}'}(d) = a$
    - $Q^{\mathcal{I}'} = \{\langle a, a, b\rangle, \langle b, a, d\rangle, \langle d, d, d\rangle, \langle a, d, d\rangle\}$
  - Do the following hold?
    1. $\forall x(P(x) \vee P(f(x)))$
    8. $\exists x f(f(f(x))) = x$
    9. $\exists x \neg Q(x, x, x) \rightarrow f(x) = t$
    10. $\exists x(Q(x, x, x) \wedge Q(c, t, x))$
- **Exercise 6.4.1:** 1-5
  - Which of the following are valid:
    1. $\forall x \forall y(R(x, y) \vee R(y, x))$
    2. $\exists x(P(x) \wedge Q(x)) \rightarrow (\exists x P(x) \wedge \exists x Q(x))$
    3. $(\exists x P(x) \wedge \exists x Q(x)) \rightarrow \exists x(P(x) \wedge Q(x))$
    4. $(\forall x P(x) \wedge \forall x Q(x)) \rightarrow \forall x(P(x) \wedge Q(x))$
    5. $\forall x(P(x) \vee Q(x)) \leftrightarrow (\forall x P(x) \vee \forall x Q(x))$

- **Bonus:**
  - Here's another model:
    - 

  - Is she valid?
- **Bonus #2:**
  - Find a first order formula that is valid in all and only the models with domains consisting of exactly two elements.
- **Extra:**
  - Is $\neg\forall x P(x) \to \neg P(x)$ valid?

Solutions:

- **Exercise 6.3.3:** 1 and 8-10
  - Here's a model over $D = \{a, b, c, d\}$:
    - $c^{\mathcal{I}} = c$ and $t^{\mathcal{I}} = a$.
    - $P^{\mathcal{I}} = \{a, b, c\}$
    - $f^{\mathcal{I}}(a) = b,\ f^{\mathcal{I}}(b) = b,\ f^{\mathcal{I}}(c) = a,\ f^{\mathcal{I}}(d) = a$
    - $Q^{\mathcal{I}} = \{\langle a, a, b\rangle, \langle b, a, d\rangle, \langle d, d, d\rangle, \langle c, a, d\rangle, \langle a, d, d\rangle\}$
  - Do the following hold?
    - 1. $\forall x(P(x) \vee P(f(x)))$
      8. $\exists x f(f(f(x))) = x$
      9. $\exists x \neg Q(x, x, x) \to f(x) = t$
      10. $\exists x(Q(x, x, x) \wedge Q(c, t, x))$
  - Answers:
    - 1. Holds
    - 8. Holds (e.g. b)
    - 9. Holds
    - 10. Holds
- **Exercise 6.3.4:** 1 and 8-10
  - Here's another model over $D = \{a, b, c, d\}$:
    - $c^{\mathcal{I}'} = b$ and $t^{\mathcal{I}'} = d$.
    - $P^{\mathcal{I}'} = \{a, c, d\}$
    - $f^{\mathcal{I}'}(a) = f^{\mathcal{I}'}(b) = f^{\mathcal{I}'}(c) = d,\ f^{\mathcal{I}'}(d) = a$
    - $Q^{\mathcal{I}'} = \{\langle a, a, b\rangle, \langle b, a, d\rangle, \langle d, d, d\rangle, \langle a, d, d\rangle\}$
  - Do the following hold?
    1. $\forall x(P(x) \vee P(f(x)))$

8. $\exists x f(f(f(x))) = x$

9. $\exists x \neg Q(x, x, x) \rightarrow f(x) = t$

10. $\exists x (Q(x, x, x) \wedge Q(c, t, x))$

- Answers:
  - 1. Holds
  - 8. Doesn't hold
  - 9. Holds
  - 10. Doesn't hold

- **Exercise 6.4.1:** 1-5
  - Which of the following are valid:

1. $\forall x \forall y (R(x, y) \vee R(y, x))$

2. $\exists x (P(x) \wedge Q(x)) \rightarrow (\exists x P(x) \wedge \exists x Q(x))$

3. $(\exists x P(x) \wedge \exists x Q(x)) \rightarrow \exists x (P(x) \wedge Q(x))$

4. $(\forall x P(x) \wedge \forall x Q(x)) \rightarrow \forall x (P(x) \wedge Q(x))$

5. $\forall x (P(x) \vee Q(x)) \leftrightarrow (\forall x P(x) \vee \forall x Q(x))$

  - Answers:
    - 1. Not valid – Counter-example:
      - $D = \{a, b\}$, $P^I = \{\}$
    - 2. Valid.
      - Because if there is some x such that $P(x)$ and $Q(x)$, then obviously there is some x such that $P(x)$ and obviously there is some x such that $Q(x)$
    - 3. Not valid – Counter-example:
      - $D = \{a, b\}$, $P^I = \{a\}$, $Q^I = \{b\}$
    - 4. Valid.
      - Because if $P(x)$ is true of every x and $Q(x)$ is true of every x, then obviously $P(x)$ and $Q(x)$ is true of every x – duh
    - 5. Valid.
      - Can't be bothered explaining this one… and don't put that in your damn homework…

- **Bonus:**
  - Here's another model…
  - Is she valid?
    - Yes. Everyone is valid.

- **Bonus #2:**
  - Find a first order formula that is valid in all and only the models with domains consisting of exactly two elements.
    - $\exists x \, \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$

- **Extra:**
  - Is $\neg \forall x P(x) \rightarrow \neg P(x)$ valid?
    - No - counter-example:
      - $D = \{a, b\}$, $P^I = \{b\}$, $g(x) = b$
      - The antecedent is true because $a \notin P^I$, so $M \vDash \neg \forall x P(x)$, but the consequent is false *on some assignment g* because $M \nvDash \neg P(x)[g]$