Handout 8

John Wilcox

Things you need to know:

- Equality rules:
  - Equality introduction:

$$\begin{array}{l|ll} \vdots & \vdots & \\ n & t = t & =\text{I} \end{array}$$

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  - Equality elimination:
    - Bilateral

$$\begin{array}{l|ll} \vdots & \vdots & \\ i & t_1 = t_2 & \\ \vdots & \vdots & \\ j & \varphi(t_1) & \\ \vdots & \vdots & \\ k & \varphi(t_2) & =\text{E}, i, j \end{array}$$

    -
- Universal quantifier rules:
  - Universal elimination:
    - If *t* contains no variable that would become bound in φ, then:

$$\begin{array}{l|ll} \vdots & \vdots & \\ i & \forall x \varphi(x) & \\ \vdots & \vdots & \\ j & \varphi(t) & \forall\text{E}, i \end{array}$$

    -
  - Universal introduction:

$$\begin{array}{l|ll} \vdots & \vdots & \\ i & \quad \boxed{c} & \\ \vdots & \quad \vdots & \\ j & \quad \varphi(c) & \\ k & \forall x \varphi(x) & \forall\text{I}, i\text{–}j \end{array}$$

    -
- Existential quantifier rules:
  - Existential elimination:

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  -
    - $\vdots$ | $\vdots$
      i | $\exists x \varphi(x)$
      $\vdots$ | $\vdots$
      j | | $\boxed{c}$ $\varphi(c)$
      $\vdots$ | | $\vdots$
      k | | ψ
      l | ψ $\exists$E, i, j–k
  - Existential introduction:
    - $\vdots$ | $\vdots$
      i | $\varphi(t)$
      $\vdots$ | $\vdots$
      j | $\exists x \varphi(x)$ $\exists$I, i

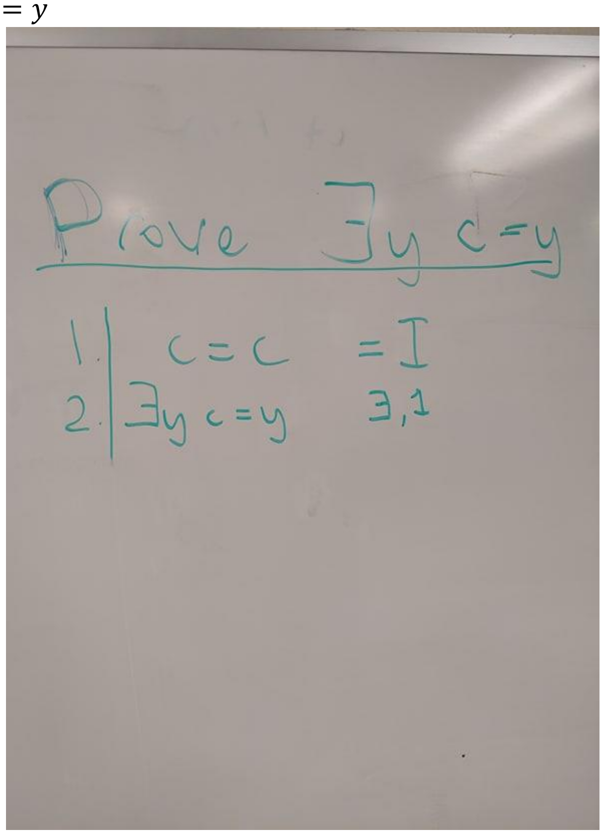
Practice exercises:

- Can two constants have the same interpretation?
- Does the following entailment hold? Justify your answer with an informal argument:
  - $\{\forall x\, \forall y\, (f(x) = f(y) \rightarrow x = y)\} \Vdash \forall x \exists y f(y) = x$
- Show that the following are derivable:
  - $\vdash \exists y\; c = y$
  - $\vdash \forall x \forall y \forall z((f(x) = y \wedge y = z) \rightarrow f(x) = z)$
  - $\{(\forall x P(x) \wedge \forall x Q(x))\} \vdash \forall x(P(x) \wedge Q(x))$

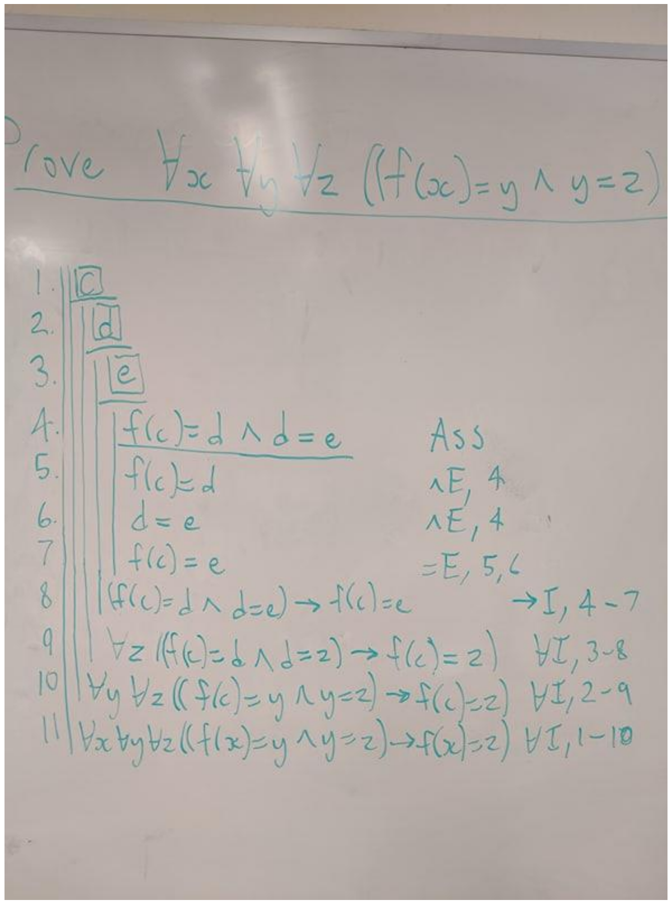
Practice solutions:

- Can two constants have the same interpretation?
  - Answer:
    - Yes.
- Does the following entailment hold? Justify your answer with an informal argument:
  - $\{\forall x\, \forall y\, (f(x) = f(y) \rightarrow x = y)\} \Vdash \forall x \exists y f(y) = x$
  - Solution:
    - $\{\forall x\, \forall y\, (f(x) = f(y) \rightarrow x = y)\}$ is true just in case for any domain objects d_i, $d_j \in D$, if $f^I(d_i) = f^I(d_j)$, then $d_i = d_j$—in other words, f is an injective function mapping each object to some object that has not already been mapped to by f.
    - $\forall x \exists y f(y) = x$ is true just in case for every domain objects $d_i \in D$, there is some domain object $d_j \in D$ such that $f^I(d_j) = d_i$—or, in other words, f is a surjective function insuring that each element of the domain is mapped to by some element.
    - If the domain is of a finite size *n*, then the first statement entails the second because *n* objects can be mapped to *n* distinct objects (i.e. f can be injective) only if each of the *n* objects in the domain is mapped to by an object (i.e. f is surjective).

- If the domain is countably infinite, then the first statement does not entail the second because we could construct a counter example where we arbitrarily label each domain object with a number $\{d_1, d_2, \dots\} = D$, and we then make it so that $f^I(d_n) = d_{2n}$. Then, every object will be mapped to some distinct object, but every odd numbered domain element will not be mapped to; for example, d_3 will not be mapped to since there is no integer z such that that $2z = 3$ and $f^I(d_z) = d_{2n}$.

- Show that the following are derivable:
  - $\vdash \exists y\, c = y$
    - 

  - $\vdash \forall x \forall y \forall z((f(x) = y \land y = z) \to f(x) = z)$



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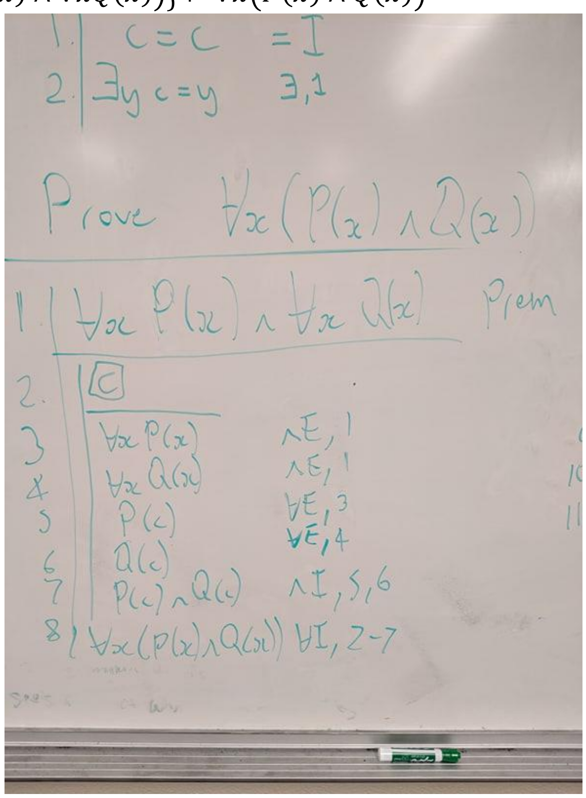
- $\{(\forall x P(x) \land \forall x Q(x))\} \vdash \forall x(P(x) \land Q(x))$



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