Handout 9

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Things you need to know:

- Decidability:
  - A yes/no question is decidable just in case there is a computation or alogirthm to answer the question
- Undecidability of FOL:
  - Take the question: is some first-order formula φ a validity?
  - It's undecidable…
- Moving on…
  - Axioms of Peano arithmetic:
    - 1. $\forall x(\neg x + 1 = 0)$
      2. $\forall x \forall y(x + 1 = y + 1 \rightarrow x = y)$
      3. $\forall x(x + 0 = x)$
      4. $\forall x \forall y\big(x + (y + 1) = (x + y) + 1\big)$
      5. $\forall x(x * 0 = 0)$
      6. $\forall x \forall y\big(x * (y + 1) = (x * y) + x\big)$
  - Induction rule:
    - $$\begin{array}{l|l}
\vdots & \vdots \\
i & \varphi(0) \\
\vdots & \vdots \\
j & \quad \boxed{n}\ \varphi(n) \\
\vdots & \quad \vdots \\
k & \quad \varphi(n+1) \\
l & \forall x \varphi(x) \qquad \text{IND}, i, j\text{–}k
\end{array}$$

Practice exercises:

- Suppose φ is a validity. Is a negated validity $\neg\varphi$ a contradiction?
- Is such a contradiction $\neg\varphi$ true in some model—i.e. satisfiable?
- If we could tell whether a formula is satisfiable, could we tell whether it is a contradiction?
- When doing your homework, can you use the above facts and the undecidability of the validity problem in your proof?
- Should you do this?
- Complete the following cases in the argument for soundness:
  - Universal elimination:
    - $$\begin{array}{l|l}
\vdots & \vdots \\
i & \forall x \varphi(x) \\
\vdots & \vdots \\
j & \varphi(t) \qquad \forall\text{E}, i
\end{array}$$

- Existential introduction:

⋮	⋮	
i	$\varphi(t)$	
⋮	⋮	
j	$\exists x \varphi(x)$	$\exists$I, i

Practice solutions:

- Suppose φ is a validity. Is a negated validity $\neg\varphi$ a contradiction?
  - Yes
- Is such a contradiction $\neg\varphi$ true in some model—i.e. satisfiable?
  - No
- If we could tell whether a formula is satisfiable, could we tell whether it is a contradiction?
  - Yes
- When doing your homework, can you use the above facts and the undecidability of the validity problem in your proof?
  - Yes
- Should you do this?
  - Yes
- Complete the following cases in the argument for soundness:
  - Definitions:
    - Soundness = $\Gamma \vdash \varphi$ implies $\Gamma \Vdash \varphi$
    - Φ_n = the set of all formulas that appear as premises along the main column or in sub proofs that are currently active in rows above row *n*
    - Property * = For any formula φ_n at row n, $\Phi_n \Vdash \varphi_n$
  - Proof strategy:
    - If we prove the property that $\Phi_n \Vdash \varphi_n$, then soundness holds
    - So we want to prove that $\Phi_n \Vdash \varphi_n$ does hold
    - So… now suppose that it doesn't hold
    - Then, there is some first place *n* where the preceding formulas follow from their respective sets Φ_k on lines $k < n$, but φ_n does not follow from Φ_n
    - We want to show this cannot happen—that this leads to a contradiction
    - To do that, we go through each of the proof rules to show it cannot happen
  - Universal elimination:

⋮	⋮	
i	$\forall x \varphi(x)$	
⋮	⋮	
j	$\varphi(t)$	$\forall$E, i

    - Proof:
      - By assumption, we suppose that $\Phi_i \Vdash \forall x \varphi(x)$, but not $\Phi_j \Vdash \varphi(t)$
      - Since $\Phi_i \Vdash \forall x \varphi(x)$, for any model *M* and variable assignment *g*, for all $d \in D, \Phi_i \Vdash_M \varphi(x)[g_d^x]$
      - Because of this, $\Phi_i \Vdash_M \varphi(x)[g_d^x]$ is true for the *d* such that $I(t) = d$

        - Then, for any model M and variable assignment g, $\Phi_i \Vdash_M \varphi(t)$ must also be true—i.e. $\Phi_i \Vdash \varphi(t)$
        - Since $\Phi_i \subseteq \Phi_j$, it follows that $\Phi_j \Vdash \varphi(t)$
        - But this contradicts the earlier supposition that it's not the case that $\Phi_j \Vdash \varphi(t)$
        - Therefore, we have a contradiction, completing an important part of the proof strategy which we outlined above
- Existential introduction:
⋮	⋮	
i	$\varphi(t)$	
⋮	⋮	
j	$\exists x \varphi(x)$	$\exists$I, i
    - Proof:
        - By assumption, we suppose that $\Phi_i \Vdash \varphi(t)$, but not $\Phi_j \Vdash \exists x \varphi(x)$
        - Since $\Phi_i \Vdash \varphi(t)$, for any model M and variable assignment g, $\Phi_i \Vdash_M \varphi(t)[g]$
        - Now for any model M and variable assignment g, there is some $d \in D$ such that $I(t) = d$
        - Because of this, for any model M and variable assignment g, $\Phi_i \Vdash_M \varphi(x)[g_d^x]$ is true for some $d \in D$, namely, the d such that $I(t) = d$
        - Then, for any model M and variable assignment g, there is some $d \in D$, such that $\Phi_i \Vdash_M \varphi(x)[g_d^x]$
        - By the definition of the semantics for existential quantifiers, for any model M, $\Phi_i \Vdash_M \exists x \varphi(x)$ and hence $\Phi_i \Vdash \exists x \varphi(x)$
        - Since $\Phi_i \subseteq \Phi_j$, it follows that $\Phi_j \Vdash \exists x \varphi(x)$
        - But this contradicts the earlier supposition that it's not the case that $\Phi_j \Vdash \exists x \varphi(x)$
        - Therefore, we have a contradiction, completing another important part of the proof strategy which we outlined above