

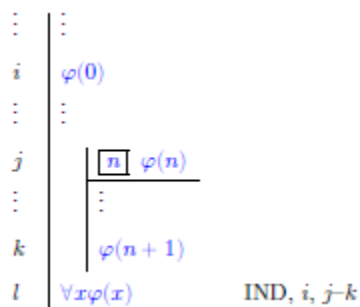
## Handout 9

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Things you need to know:

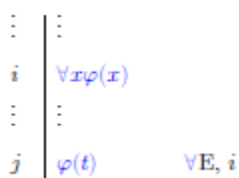
- Decidability:
  - o A yes/no question is decidable just in case there is a computation or algorithm to answer the question
- Undecidability of FOL:
  - o Take the question: is some first-order formula  $\varphi$  a validity?
  - o It's undecidable...
- Moving on...
  - o Axioms of Peano arithmetic:
    1.  $\forall x(-x + 1 = 0)$
    2.  $\forall x\forall y(x + 1 = y + 1 \rightarrow x = y)$
    3.  $\forall x(x + 0 = x)$
    4.  $\forall x\forall y(x + (y + 1) = (x + y) + 1)$
    5.  $\forall x(x * 0 = 0)$
    6.  $\forall x\forall y(x * (y + 1) = (x * y) + x)$

- o Induction rule:

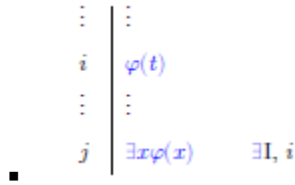


Practice exercises:

- Suppose  $\varphi$  is a validity. Is a negated validity  $\neg\varphi$  a contradiction?
- Is such a contradiction  $\neg\varphi$  true in some model—i.e. satisfiable?
- If we could tell whether a formula is satisfiable, could we tell whether it is a contradiction?
- When doing your homework, can you use the above facts and the undecidability of the validity problem in your proof?
- Should you do this?
- Complete the following cases in the argument for soundness:
  - o Universal elimination:

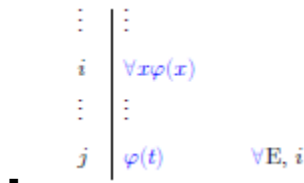


- o Existential introduction:



Practice solutions:

- Suppose φ is a validity. Is a negated validity ¬φ a contradiction?
  - o Yes
- Is such a contradiction ¬φ true in some model—i.e. satisfiable?
  - o No
- If we could tell whether a formula is satisfiable, could we tell whether it is a contradiction?
  - o Yes
- When doing your homework, can you use the above facts and the undecidability of the validity problem in your proof?
  - o Yes
- Should you do this?
  - o Yes
- Complete the following cases in the argument for soundness:
  - o Definitions:
    - Soundness = Γ ⊢ φ implies Γ ⊨ φ
    - Φ<sub>n</sub> = the set of all formulas that appear as premises along the main column or in sub proofs that are currently active in rows above row n
    - Property \* = For any formula φ<sub>n</sub> at row n, Φ<sub>n</sub> ⊨ φ<sub>n</sub>
  - o Proof strategy:
    - If we prove the property that Φ<sub>n</sub> ⊨ φ<sub>n</sub>, then soundness holds
    - So we want to prove that Φ<sub>n</sub> ⊨ φ<sub>n</sub> does hold
    - So... now suppose that it doesn't hold
    - Then, there is some first place n where the preceding formulas follow from their respective sets Φ<sub>k</sub> on lines k < n, but φ<sub>n</sub> does not follow from Φ<sub>n</sub>
    - We want to show this cannot happen—that this leads to a contradiction
    - To do that, we go through each of the proof rules to show it cannot happen
  - o Universal elimination:



- Proof:
  - By assumption, we suppose that Φ<sub>i</sub> ⊨ ∀xφ(x), but not Φ<sub>j</sub> ⊨ φ(t)
  - Since Φ<sub>i</sub> ⊨ ∀xφ(x), for any model M and variable assignment g, for all d ∈ D, Φ<sub>i</sub> ⊨<sub>M</sub> φ(x)[g<sub>d</sub><sup>x</sup>]
  - Because of this, Φ<sub>i</sub> ⊨<sub>M</sub> φ(x)[g<sub>d</sub><sup>x</sup>] is true for the d such that I(t) = d

- Then, for any model  $M$  and variable assignment  $g$ ,  $\Phi_i \Vdash_M \varphi(t)$  must also be true—i.e.  $\Phi_i \Vdash \varphi(t)$
  - Since  $\Phi_i \subseteq \Phi_j$ , it follows that  $\Phi_j \Vdash \varphi(t)$
  - But this contradicts the earlier supposition that it's not the case that  $\Phi_j \Vdash \varphi(t)$
  - Therefore, we have a contradiction, completing an important part of the proof strategy which we outlined above
- Existential introduction:

⋮	⋮	
i	⋮	
⋮	⋮	
j	⋮	
		$\exists x\varphi(x) \quad \exists i, i$

▪ Proof:

- By assumption, we suppose that  $\Phi_i \Vdash \varphi(t)$ , but not  $\Phi_j \Vdash \exists x\varphi(x)$
- Since  $\Phi_i \Vdash \varphi(t)$ , for any model  $M$  and variable assignment  $g$ ,  $\Phi_i \Vdash_M \varphi(t)[g]$
- Now for any model  $M$  and variable assignment  $g$ , there is some  $d \in D$  such that  $I(t) = d$
- Because of this, for any model  $M$  and variable assignment  $g$ ,  $\Phi_i \Vdash_M \varphi(x)[g_d^x]$  is true for some  $d \in D$ , namely, the  $d$  such that  $I(t) = d$
- Then, for any model  $M$  and variable assignment  $g$ , there is some  $d \in D$ , such that  $\Phi_i \Vdash_M \varphi(x)[g_d^x]$
- By the definition of the semantics for existential quantifiers, for any model  $M$ ,  $\Phi_i \Vdash_M \exists x\varphi(x)$  and hence  $\Phi_i \Vdash \exists x\varphi(x)$
- Since  $\Phi_i \subseteq \Phi_j$ , it follows that  $\Phi_j \Vdash \exists x\varphi(x)$
- But this contradicts the earlier supposition that it's not the case that  $\Phi_j \Vdash \exists x\varphi(x)$
- Therefore, we have a contradiction, completing another important part of the proof strategy which we outlined above