Phil 150: Section 1 Handout

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Utility of logic:
1. Conceptual clarity
2. Argumentative rigor
3. Everyday benefits:
    - Clearer reasoning
    - Avoiding pointless arguments
    - Etc…

General tips for succeeding in logic:
1. Fight the initial fear
2. Learn definitions **well**:
    a. Memorize definitions ***verbatim*-**ish
    b. Have facility with examples
3. Break things down **LOTS**:
    a. Don't be daunted by first appearances
    b. Break things down using definitions
4. Utilize various proof strategies:
    a. Talk about some "arbitrary" member or set
        i. To prove relationships between sets (e.g. subsethood, identity, etc.), talk about the features of some arbitrary set or some arbitrary member of the set
    b. Reason by cases
        i. Break things down into cases (e.g. "either this thing is true or that thing is true") and show that you can reach the same conclusion in all cases
    c. Proving equivalences
        i. For sets: prove one is a subset of another, and vice versa
    d. Conditional proofs
        i. To prove something q follows from another thing p (i.e. a conditional $p \rightarrow q$), assume p and show that q follows
    e. Contraposition
        i. To prove a conditional of the form $p \rightarrow q$, prove $\neg q \rightarrow \neg p$—that is, to prove that "if p is the case, then q is the case", show that "if q is not the case, then p is not the case"
    f. Reductio ad absurdum
        i. To prove p for any p, assume $\neg p$ and show that that results in a contradiction
5. Practice! Practice! Practice!

[1] Footnote: New Zealand is not Australia

Some terms and definitions you need to know:

- Basic building blocks:
  - Elements *a, b, c…*
  - Sets $\{a, b, \ldots, \{a,b\}\}$
  - $x \in A$ = element *x* is in set *A*
  - $x \notin A$ = element *x* is *not* in set *A*
  - $x \triangleq y$ means that *x is* ***defined*** *to be* y
- Definitions of a set
  - $A \triangleq \{x|\ so-and-so\}$ means that A is *defined* to be the set of all things *x* such that *x* meets condition "so-and-so"
- Subsethood:
  - $A \subseteq B$ means that for every element $a \in A$, it is also the case that $a \in B$
- Set identity:
  - $A = B \triangleq (A \subseteq B\ and\ B \subseteq A)$
- Union:
  - $A \cup B \triangleq \{x|x \in A\ or\ x \in B\}$
- Intersection:
  - $A \cap B \triangleq \{x|x \in A\ and\ x \in B\}$
- Complement:
  - $A - B \triangleq \{x|\ x \in A\ and\ x \notin B\}$
- Power set:
  - $\rho(A) \triangleq \{B|B \subseteq A\}$
- Ordered pair:
  - $< a, b > \triangleq$ an object which has *a* as its first element and *b* as its second **(Order matters!)**
- Cartesian product:
  - $A \times B \triangleq \{< a, b > |\ a \in A\ and\ b \in B\}$
- Binary relation *R* on *A:*
  - Some subset ordered pairs $R \subseteq A \times A$
- Properties of relations:
  - *R* is reflexive if:
    - *Raa* for all $a \in A$
  - *R* is symmetric if:
    - Whenever *Rab*, then *Rba* is also true for any $a, b \in A$
  - *R* transitive if:
    - Whenever *Rab* and *Rbc*, then *Rac* for any $\mathrm{a, b, c} \in \mathrm{A}$
  - *R* Euclidean if:
    - Whenever *Rab* and *Rac*, then *Rbc* for any $\mathrm{a, b, c} \in \mathrm{A}$
- A relation $R \subseteq A \times B$ is functional if:
  - For every $a \in A$, there is at most one $b \in B$ such that *Rab*
  - *A* is called the **domain** of the function
  - *B* is called the **range** of the function
  - *A* and *B* may or may not be distinct
- A functional relation (i.e. a function) is injective if:
  - Every distinct element of *A* is mapped to some distinct element of *B*
- A functional relation (i.e. a function) is surjective if:
  - Everything in the *range* (the set that is mapped to—in this case, *B*) is mapped to by something in the domain

Practice exercises **(solutions are below)**:

1. $A - B \subseteq (A \cup C) - B$
2. $A \cup B \subseteq (A \cup C) \cup (B \cup C)$
3. $A \times B \subseteq A \times (B \cup C)$
4. Relations:
    a. Suppose we have three elements in a set $A = \{a, b, c\}$, and a relation *R* defined on this set
    b. Suppose *R* is symmetric and that $< a, b >, < b, c >, < c, a > \in R$—that is, Rab, Rbc, Rca
    c. Prove that *R* defined on this set is transitive
5. Suppose we have a function $F: A \rightarrow \rho(A)$ such that $F(a) = \{a\}$ for any $a \in A$
    a. Is *F* necessarily one-to-one?
    b. Is *F* necessarily onto?

Powersets:

- Definition:
    - $\rho(A) \triangleq \{B | B \subseteq A\}$
- Example:
    - Suppose $A = \{a,$
$b,$
$\{c, d\}\}$
    - Then, the powerset $\rho(A) = \{\{\},$
$\{a\},$
$\{b\},$
$\{a, b\},$
$\{\{c, d\}\}$
$\{a, \{c, d\}\}$
$\{b, \{c, d\}\}$
$\{a, b, \{c, d\}\}$
- Common confusions:
    - Elements *are* sometimes sets and sometimes *not* sets:
        - For *A* above, *a* is an element that isn't a set, but $\{c, d\}$ is an element that is a set
    - If a set has an element which is itself a set, then the elements of the latter set are not necessarily elements of the former
        - Consider *A* = $\{a, b, \{c, d\}\}$. $\{c, d\}$ is an element of *A*. But *c* is not an element of *A*. Instead, *c* is an *element of an element* of *A*.
    - If a set *B* is an element of *A*, then that set is *not* in the power set, but the *set containing that set is*
        - E.g. $\{c, d\}$ is not in the power set because it is not a subset of elements from *A*, but $\{\{c, d\}\}$ is in the power set because it is a subset of elements from *A*, namely, the subset containing the element $\{c, d\}$

Solutions:

1. $A - B \subseteq (A \cup C) - B$ **(considering some arbitrary element, conditional proof)**
    a. Consider some arbitrary element a **(note: considering some arbitrary element)**
    b. Suppose $a \in A - Ba$ **(note: conditional proof)**
        i. Then, by the definition of complement, $a \in A$ and $a \notin B$ **(note: this shows that you know the definition explicitly)**
        ii. Now, $a \in (A \cup C) - B$ just in case $a \in (A \cup C)$ and $a \notin B$
        iii. By the definition of union, $a \in (A \cup C)$ just in case $a \in A$ or $a \in C$ (or both)
        iv. $a \in (A \cup C)$, since $a \in A$
        v. And since, as mentioned earlier, $a \notin B$, it follows that $a \in (A \cup C) - B$.

c. Hence, every member of $A - B$ is a member of $(A \cup C) - B$, so $A - B \subseteq (A \cup C) - B$

2. $A \cup B \subseteq (A \cup C) \cup (B \cup C)$ **(proof by cases)**
   - a. Consider some arbitrary element $a \in A \cup B$
   - b. Then, by definition, either $a \in A$ or $a \in B$ **(note: proof by cases)**
     - i. Suppose $a \in A$
       - 1. Now, $a \in (A \cup C)$ because $a \in A$
       - 2. Therefore, $a \in (A \cup C) \cup (B \cup C)$ because $a \in (A \cup C)$
     - ii. Suppose $a \in B$
       - 1. Now, $a \in (B \cup C)$ because $a \in B$
       - 2. Therefore, $a \in (A \cup C) \cup (B \cup C)$ because $a \in (B \cup C)$
   - c. In both cases, $a \in (A \cup C) \cup (B \cup C)$ and so $A \cup B \subseteq (A \cup C) \cup (B \cup C)$
3. $A \times B \subseteq A \times (B \cup C)$
   - a. Consider some arbitrary member $< a , b > \in A \times B$
   - b. Then, by definition of the cross-product, $a \in A$ and $b \in B$
   - c. By the same definition, $< a, b > \in A \times (B \cup C)$ just in case $a \in A$ and $b \in (B \cup C)$
   - d. We already know that $a \in A$, and we also know that $b \in (B \cup C)$ since $b \in B$
   - e. Therefore, $< a, b > \in A \times (B \cup C)$ and so every member of $A \times B$ is a member of $A \times (B \cup C)$—that is, $A \times B \subseteq A \times (B \cup C)$
4. Relations:
   - a. Question:
     - i. Suppose we have three elements in a set $A = \{a, b, c\}$, and a relation *R* defined on this set
     - ii. Suppose *R* is symmetric and that $< a, b >, < b, c >, < c, a > \in R$—that is, Rab, Rbc, Rca
     - iii. Prove that *R* defined on this set is transitive
   - b. Proof:
     - i. Since *R* is symmetric, by definition, if Rxy, then Ryx for any elements *x, y*
     - ii. Since Rca, and since *R* is symmetric, it follows that Rac
     - iii. Now, by definition, *R* is transitive just in case whenever Rxy, and Ryz, then Rxz for any elements *x, y, z*
     - iv. Now Rab and Rbc, and, as just proved, Rac
     - v. Since this satisfies transitivity, and since no other relations violate transitivity, we can conclude that *R* is transitive
5. Suppose we have a function $F: A \rightarrow \rho(A)$ such that $F(a) = \{a\}$ for any $a \in A$
   - a. Is *F* necessarily one to one?
     - i. By definition, *F* would be one-to-one just in case every element in A is mapped to a **<u>distinct</u>** element in $\rho(A)$ (so every element of *A* has to have a mapping, and the mapping can't map two elements of *A* to the same thing)
     - ii. Now, *F* is necessarily one-to-one since every element is mapped to one distinct thing—the set containing itself
   - b. Is *F* necessarily onto?
     - i. By definition, *F* is onto just in case every element in $\rho(A)$ is mapped to by *F*
     - ii. However, it is not necessarily onto, since we can construct a counter example:
       - 1. Suppose $A = \{a, b\}$
       - 2. Then, $\{a, b\} \in \rho(A)$ (since $\{a, b\}$ is a subset of $A = \{a, b\}$ in the sense that every element in $\{a, b\}$ is also in *A*)
       - 3. But $\{a, b\}$ is not mapped to, since "a, b" is not an element in the set *A* (even though *a* and *b* are *separately* elements that are mapped to $\{a\}$ and $\{b\}$ respectively)