Phil 150: Section 2 Handout

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Some definitions you need to know:
- Negation:
  - Informal reading: "A is not the case"

A	$(\neg A)$
T	F
F	T

- Conjunction:
  - Informal reading: "A is true and B is true"

B	A	$(A \wedge B)$
T	T	T
T	F	F
F	T	F
F	F	F

- Disjunction:
  - Informal reading: "Either A is true, B is true, or both are true"

B	A	$(A \vee B)$
T	T	T
T	F	T
F	T	T
F	F	F

- Material implication:
  - Informal reading: *Kind of* like "If A is true, then B is true"
  - Note! Material conditional departs from out intuitive notions of "If…, then …" statements in many ways

B	A	$(A \to B)$
T	T	T
T	F	T
F	T	F
F	F	T

[1] Footnote: New Zealand is still not Australia.

- Valuation:
  - Let $Prop$ be the set of all propositional atoms $A_1, A_2, \ldots, A_n$
  - A valuation $v \triangleq$ a function $v: Prop \rightarrow \{T, F\}$ assigning T or F to every propositional atom in *Prop*
- Validity:
  - A formula φ is logically valid ≜ for every valuation *v*, $v \vDash \varphi$—that is, the valuation makes the formula true
- Contradiction:
  - A formula φ is logically valid ≜ for every valuation *v*, $v \nvDash \varphi$—that is, the valuation assigns *F* to φ
- Satisfiability:
  - A formula φ is satisfiable ≜ there is at least one valuation such that $v \vDash \varphi$
  - A set of formulas Γ is satisfiable ≜ there is at least one valuation such that $v \vDash \varphi$ for every $\varphi \in \Gamma$
- Entailment:
  - Where Γ is a set of formulas, Γ logically entails φ (written $\Gamma \Vdash \varphi$) ≜ for every valuation such that $v \vDash \gamma$ for every formula $\gamma \in \Gamma$, it is also the case that $v \vDash \varphi$
- Equivalence:
  - Formulas γ and φ are equivalent ≜ $\gamma \Vdash \varphi$ and $\varphi \Vdash \gamma$ (i.e. they always have the same truth-values)
- Truth-functional completeness:
  - A set of connectives ***C*** is truth-functionally complete ≜ every other *n*-ary connective (for any *n*) can be defined out of connectives in *C*.

Practice exercises:

1. Is this formula valid?
   a. $((A \vee B) \vee \neg A)$
   b. $(A \wedge \neg A)$
2. Are any of the above two formulas satisfiable?
3. Does the following entailment hold?
   a. $\{(\neg A \vee B), \neg B\} \Vdash (\neg A)$
4. Does the following equivalence hold?
   a. $(\neg(\neg A \wedge B)) \dashv\Vdash (A \vee \neg B)$[2]
5. Is this valid? Show why or why not.
   a. $((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A))$
6. Does the following hold?
   a. $\{(A \rightarrow B), \neg B\} \Vdash (\neg A)$
7. Suppose this corollary is true:
   a. Corollary 642: $\{\vee, \neg\}$ is truth-functionally complete.
   b. Now show that $\{\wedge, \neg\}$ is also truth functionally complete

[2] Sorry folks. Can't figure out how to do this thing ⊩ backwards, so excuse the mismatch between my notation and the course reader's ¯_(ツ)_/¯

Practice exercise solutions:

1. Is this formula valid?
    a. $((A \vee B) \vee \neg A)$
        i. Yes. Every valuation makes it true.

A	B	$((A$	$\vee$	$B)$	$\vee$	$\neg$	$A)$
T	T	T	T	T	T	F	T
T	F	T	T	F	T	F	T
F	T	F	T	T	T	T	F
F	F	F	F	F	T	T	F

    b. $(A \wedge \neg A)$
        i. No, at least one valuation does not make it true. (You can give the truth table, as above.)
2. Are any of the above two formulas satisfiable?
    a. Yes. $((A \vee B) \vee \neg A)$ is satisfiable because there is at least one valuation where it's true.
3. Does the following entailment hold?
    a. $\{(\neg A \vee B), \neg B\} \Vdash (\neg A)$
        i. Yes. Every valuation where the premises are true is one where the conclusion is true.

A	B		$(\neg$	A	$\vee$	$B)$	$(\neg$	$B)$	$(\neg$	$A)$
T	T		F	T	T	T	F	T	F	T
T	F		F	T	F	F	T	F	F	T
F	T		T	F	T	T	F	T	T	F
F	F		T	F	T	F	T	F	T	F

4. Does the following equivalence hold?
    a. $(\neg(\neg A \wedge B)) \dashv\Vdash (A \vee \neg B)$
        i. Yes. (You can give the truth table, as above.)
5. Is this valid? Show why or why not.
    a. $((A \to B) \to (\neg B \to \neg A))$
        i. It's valid. (You can give the truth table, as above, to show this.)
6. Does the following hold?
    a. $\{(A \to B), \neg B\} \Vdash (\neg A)$
        i. Yes. (You can give the truth table, as above.)
7. Suppose this corollary is true:
    a. Corollary 642: $\{\vee, \neg\}$ is truth-functionally complete.
    b. Now show that $\{\wedge, \neg\}$ is also truth functionally complete
        i. By definition, a set of connectives C is truth functionally complete if every other n-ary connective (for any n) can be defined out of connectives in C.
        ii. If the connectives of C can be defined out of connectives in some other set C^*, then C^* must also be truth-functionally complete.
        iii. Consequently, $\{\wedge, \neg\}$ is truth functionally complete if we can define the connectives $\{\vee, \neg\}$ out of $\{\wedge, \neg\}$.
        iv. Consequently, we now turn to show that this can be done.
        v. Trivially, negation (i.e. $\neg$) in one set is defined the same as negation in the other set.

vi. So the real task is to show that disjunction (i.e. ∨) can be defined out of {∧, ¬}.

vii. We show this below with a truth table:

A	*B*	(*A*	∨	*B*)	¬	(¬	*A*	∧	¬	*B*)
T	*T*	*T*	*T*	*T*	*T*	*F*	*T*	*F*	*F*	*T*
T	*F*	*T*	*T*	*F*	*T*	*F*	*T*	*F*	*T*	*F*
F	*T*	*F*	*T*	*T*	*T*	*T*	*F*	*F*	*F*	*T*
F	*F*	*F*	*F*	*F*	*F*	*T*	*F*	*T*	*T*	*F*

viii. We see that they have exactly the same truth-tables, meaning that disjunction can be defined out of negation and conjunction.

ix. Thenceforth, {∧, ¬} is also truth-functionally complete