Phil 150: Section 4 Handout

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Modal logic -- things you need to know:

- Additions to the syntax:[2]
  - If φ is a wff, then so is $\Box\varphi$
  - If φ is a wff, then so is $\Diamond\varphi$
- Modifications of the semantics:[3]
  - A modal model M is a triple (S, R, V) where:
    - S is some set of points or states – intuitively "states of affairs"
    - $R \subseteq S \times S$ is a relation on S, usually called the "accessibility relation"
    - $V: Prop \rightarrow \rho(S)$ is a modal valuation, specifying which atoms are true at which points
  - For example, consider the below:

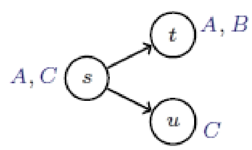

Figure 5.1: A simple model $\mathcal{M}_0 = (S_0, R_0, V_0)$

- $S_0 = \{s, t, u\}$ are the states.
- R_0st and R_0su; i.e., $R_0 = \{\langle s, t\rangle, \langle s, u\rangle\}$.
- $V_0(A) = \{s, t\}$, $V_0(B) = \{t\}$, and $V_0(C) = \{s, u\}$.

  - Semantic differences in modal logic:
    - Truth of a formula:
      - $M, s \vDash A$ iff $s \in V(A)$ where $A \in Prop$[4]
      - $M, s \vDash \Box\varphi$ iff $M, t \vDash \varphi$ for "every" $t \in S$ such that *Rst*
      - $M, s \vDash \Diamond\varphi$ iff $M, t \vDash \varphi$ for some $t \in S$ such that *Rst*
      - Everything else is the same…
    - Validity of a formula:
      - A formula is valid if $M, s \vDash \varphi$ for every model M and state s
- Modal frames:
  - Modal frame:
    - = A pair $\mathcal{F} = (S, R)$, i.e. just a model $M - V$
  - Frame validity:
    - A formula φ is valid on a frame $\mathcal{F} = (S, R)$ if it is true at every point s in every M built on $\mathcal{F}$
    - This is written as $(S, R) \vDash \varphi$ or, equivalently, $\mathcal{F} \vDash \varphi$

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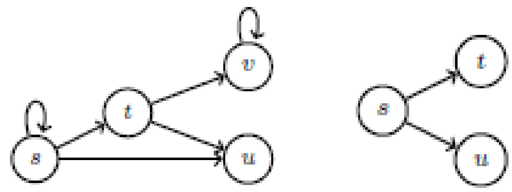
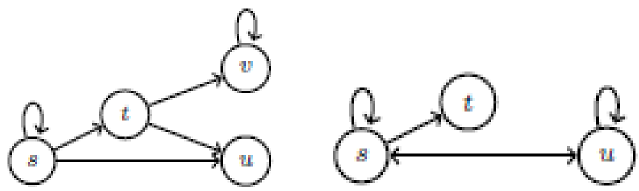
[1] Note: New Zealand isn't England either…
[2] Syntax = the rules governing the language of modal logic.
[3] Semantics = the meaning of the language, or the conditions under which formulas are true.
[4] This is different to a valuation in Boolean logic which makes a propositional atom true just in case that atom is sent to T instead of F.

Practice exercises:

1. Do the even numbered parts (plus part 1) of exercise 5.1.1.
   remember to explain in your homework
2. Do the even numbered parts (plus part 1) of exercise 5.1.1.
3. Find a formula that is valid in the left frame but not the right:

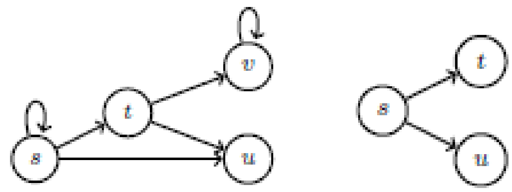




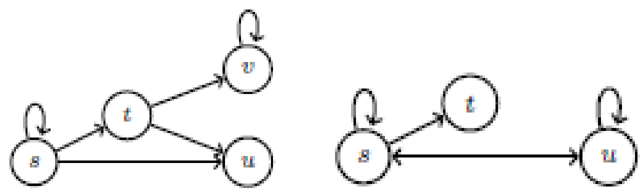
Solutions:

1. Exercise 5.1.1.
   1. Holds
      - By definition, $M_1, s \vDash \Diamond\Diamond C$ just in case there is some point *p* is accessible from *s* such that there is some point *q* accessible from *p* and *C* is true at *q*
      - We can see that $M_1, s \vDash \Diamond\Diamond C$ is true since *t* is accessible from *s*, *u* is accessible from *t* and *C* is true at *u*
   2. Doesn't hold
      - By definition, $M_1, s \vDash \Box\Box C$ just in case for every point *p* accessible from *s* and for every point *q* accessible from any such *p*, *C* is true at *q*
      - $M_1, s \nvDash \Box\Box C$ because *t* is accessible from *s*, *t* is accessible from *t*, and *C* is *not* true at *t*, thus providing a counter-example to the claim that $M_1, s \vDash \Box\Box C$
   4. Holds
   6. Holds
   8. Holds
   10. Holds
   12. Holds

2. Find a formula that is valid in the left frame but not the right:



$(S, R) \vDash \neg \diamond\diamond\top \to \neg \diamond \Box \bot$



$(S, R) \vDash \diamond \Box \bot \to \diamond\diamond (\neg\Box \bot \wedge \neg \diamond\diamond \Box \bot)$

Exercise 5.1.7:

- A relation is called *serial* if for each *s* there *s* some *t* such that *Rst.* Show that *R* is serial iff $(S, R) \vDash \Box A \to \diamond A$.

  NOTE: You can prove this result without specifying a valuation *V*, but you will need to specify a valuation for each of your 3 homework proofs… Examples of how to do this are in the reader

  - **Breaks down equivalence into component implications**
    - To show this, we need to show that the implication holds in both directions:
      1. if that *R* is serial, then $(S, R) \vDash \Box A \to \diamond A$
      2. if $(S, R) \vDash \Box A \to \diamond A$, then *R* is serial
  - **Prove first implication**
    - **Assume the antecedent**
      - Suppose *R* is serial.
    - **Apply definition to the case at hand**
      - Then, by definition, for each *s* there is some *t* such that *Rst*.
    - **Select some *s***
      - Now take some *s*, and note there is some *t* such that *Rst*
    - **Signpost the consequent you want to prove**
      - Now we hope to prove that $\Box A \to \diamond A$
    - **Suppose antecedent**
      - Now suppose $\Box A$ is true at *s*

        - **Applying definition**
            - Then, for every state *p* such that *Rsp*, A is true at *p*
            - Now t is one such state, and A is true at t
            - But since there is some *p* (namely, *t*) such that *Rsp* and *A* is true at *p*, it follows that $\diamond A$ is true at *s*.
    - **Conclude the sub-proof**
        - It follows that if that *R* is serial, then $(S, R) \vDash \Box A \rightarrow \diamond A$
- **Prove second implication**
    - **State the contrapositive**
        - To prove that if $(S, R) \vDash \Box A \rightarrow \diamond A$, then *R* is serial, we will prove the contra positive: that is, if *R* is not serial, then it is not the case that $(S, R) \vDash \Box A \rightarrow \diamond A$, so $(S, R) \nvDash \Box A \rightarrow \diamond A$
    - **Assume the antecedent**
        - So suppose *R* is not serial.
    - **Select arbitrary frame (note: you will need to select valuations too in your homework)**
        - Now consider some arbitrary frame such that *R* is non-serial, and consider some valuation V_1.
        - We hope to show that $(S_1, R_1, V_1) \nvDash \Box A \rightarrow \diamond A$—that is, given some valuation *V* and arbitrary frame (S_1, R_1, V_1), $\vDash \Box A \rightarrow \diamond A$ is false
    - **Apply the negation of the definition**
        - Recall that *R* is not serial.
        - Then, by definition, for some *s* there is no *t* such that *Rst*
    - **Select arbitrary *s***
        - Now take some such *s*, and note there is no *t* such that *Rst*
    - **Signpost the consequent you want to prove**
        - Now we hope to prove that $(S_1, R_1, V_1), s \nvDash \Box A \rightarrow \diamond A$
    - **Prove the consequent—in this case, an implication**
        - **Assume (or show) the antecedent in the implication**
            - Since no state *t* is accessible from *s*, $\Box\varphi$ will be true at *s* for any formula φ
            - Now *A* is one such φ, so *s* satisfies the antecedent of the conditional: namely, $\Box A$.
        - **Show the consequent is false**
            - However, it is not that case that $\diamond A$, for $\diamond A$ is true just in case there is some point *t* such that *Rst* and *A* is true at *t*, and this clearly not the case since there simply is no such *t* such that *Rst*
    - **Conclude the sub-proof**
        - Consequently, we have identified an arbitrary non-serial model such that $(S_1, R_1, V_1) \nvDash \Box A \rightarrow \diamond A$, and so $\Box A \rightarrow \diamond A$ is not a validity across non-serial frames
        - Therefore, if *R* is not serial, then it is not the case that $(S, R) \vDash \Box A \rightarrow \diamond A$, or, in other words, if $(S, R) \vDash \Box A \rightarrow \diamond A$, then *R* is serial.
- **Conclude:**
    - This proves that that *R* is serial iff $(S, R) \vDash \Box A \rightarrow \diamond A$.